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Applications of Partially Quenched Chiral Perturbation Theory

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ABSTRACT: Partially quenched theories are theories in which the valence- and sea-quark masses are different. In this paper we calculate the nonanalytic one-loop corrections of some physical quantities: the chiral condensate, weak decay constants, Goldstone boson masses, B_K and the $K^+ \to \pi^+\pi^0$ decay amplitude, using partially quenched chiral perturbation theory. Our results for weak decay constants and masses agree with, and generalize, results of previous work by Sharpe. We compare B_K and the K^+ decay amplitude with their real-world values in some examples. For the latter quantity, two other systematic effects that plague lattice computations, namely, finite-volume effects and unphysical values of the quark masses and pion external momenta are also considered. We find that typical one-loop corrections can be substantial.

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1. Introduction

Recently, large scale numerical lattice QCD computations have started to move away from the quenched approximation by taking virtual-quark loop effects into account. However, since the computation of quark determinants is much more expensive than that of quark propagators, one is often restricted to only very few values of the sea-quark mass, while, given an ensemble of "dynamical fermion" gauge configurations, one can explore a much larger range of values for the valence-quark masses. This naturally leads us to consider partially quenched theories, which are theories in which the valence- and sea-quark masses are not equal.

Chiral Perturbation Theory (ChPT) plays an important role in the analysis of results from lattice computations (see for instance ref. [1]). Since the quenched or partially quenched approximations of lattice QCD are theories different from the full unquenched theory, ChPT needs to be adapted to these situations. For the quenched case, this has been extensively investigated (for a rather complete list of references, see ref. [2]), but much less work has been done for the partially quenched case.

Partially Quenched Chiral Perturbation Theory (PQChPT) was developed in ref. [3], and used to calculate one-loop expressions for Goldstone boson masses and decay constants in ref. [4]. In this paper, we revisit Goldstone boson masses and decay constants, and we also give the one-loop expressions for the chiral condensate, B_K and the $K^+ \to \pi^+\pi^0$ decay amplitude. (For an application to heavy-light decay constants and B-parameters, see ref. [5].)

In the partially quenched case (unlike in the quenched case), the η' meson is heavy in the sense that its mass does not vanish in the chiral limit. One can therefore approach PQChPT in the same way as one does in the unquenched case, where the η' is integrated out, and only the Goldstone bosons are kept in the effective lagrangian. This is the approach taken

in ref. [4]. In the completely quenched case, however, the η' is light, and needs to be kept in the effective lagrangian, along with the other Goldstone bosons. The theory depends on a mass scale m_0, independent of the quark masses, which in the unquenched theory is the singlet part of the η' mass, but in the quenched theory no longer appears in any pole mass [6,7,8,9]. A systematic expansion in the quenched case only exists if we treat the ratio of m_0 to the chiral-symmetry breaking scale as an independent *small* parameter, in addition to the light quark masses, and if we stay away from the chiral limit, where infrared divergences occur [8,9,10,11].

In partially quenched lattice computations, it may happen that the scale set by the sea-quark mass is not small compared to the scale m_0. In this paper we therefore keep the η' in the effective theory, and show that one can still develop PQChPT systematically if one assumes, as in QChPT, that the ratio of m_0 to the chiral-symmetry breaking scale is reasonably small. This generalizes the results of ref. [4].

The outline of this paper is as follows. In section 2, we give a quick review of PQChPT, and discuss the role of the η' in more detail. We explain more precisely how our calculation generalizes that of ref. [4]. We then calculate in section 3 the chiral condensate and Goldstone boson masses and decay constants to one loop as a function of the quark masses, and investigate their dependence on m_0. We restrict ourselves to the theory with N degenerate sea quarks. In section 4, we calculate B_K for non-degenerate valence-quark masses and the $K^+ \to \pi^+\pi^0$ decay amplitude for degenerate valence-quark masses. For $K^+ \to \pi^+\pi^0$ we also discuss other systematic errors which have affected lattice computations of this quantity to date. In section 5, we give numerical examples of the role of one-loop corrections for B_K and $K^+ \to \pi^+\pi^0$ in a comparison between the real world and the partially quenched theory with parameters typical of a lattice computation. More details on the assumptions underlying these numerical examples will be given in due course. We end with our conclusions.

2. Essentials of Partially Quenched Chiral Perturbation Theory

Consider a QCD-like theory with n flavors of quarks q_i, each with arbitrary mass m_i. We then partially quench the theory by adding $n-N$ flavors of (unphysical) quarks $\tilde{q}_i$ obeying bosonic statistics [12], which we will call ghost-quarks. The ghost-quark masses are chosen to equal those of the first $n-N$ physical quarks. These $n-N$ quarks are then quenched since their loop contributions are exactly canceled by their bosonic counterparts. The remaining N quarks contribute through (sea-)quark loops to physical quantities. With the $n-N$ quarks identified as valence quarks, the theory corresponds to partially quenched QCD with N flavors of sea quarks. Recent partially quenched lattice computations use degenerate sea-quark masses, hence, in the following, we will set all sea-quark masses m_i, $i = n-N+1, \cdots, n$, equal to m_S.

The full chiral symmetry of the theory is the semi-direct product of graded groups $G \equiv [SU(n|n-N) \otimes SU(n|n-N)] \circledS U(1)$ after the anomaly has been taken in account [8]. We will briefly discuss the construction of the lagrangian for PQChPT for our choice of quark masses.

The unitary field Σ is defined through

$$\Sigma \equiv \exp\left(2i\Phi/f\right) \tag{1}$$

from the $(2n-N) \times (2n-N)$ hermitian matrix field

$$\Phi = \begin{pmatrix} \phi & \chi^\dagger \\ \chi & \tilde{\phi} \end{pmatrix} , \tag{2}$$

where ϕ is the $n \times n$ matrix of ordinary mesons made from the n ordinary quarks and antiquarks, $\tilde{\phi}$ is the corresponding $(n-N) \times (n-N)$ matrix for ghost-quark mesons, and χ is an $(n-N) \times n$ matrix of fermionic mesons made from a ghost-quark and an ordinary antiquark. f is the tree-level pion decay constant. The $(2n-N) \times (2n-N)$ quark-mass matrix

$\hat{M}$ is defined by $\hat{M}_{ij} = m_i \delta_{ij}$, where, as already discussed, the masses m_i for $i = 1, \cdots, n-N$ are arbitrary, and equal to the ghost-quark masses, $i = n+1, \cdots, 2n-N$. The remaining N masses m_i, $i = n-N+1, \cdots, n$, are degenerate and equal to m_S.

As discussed in refs. [8,3], the super-η' field $\Phi_0 \equiv \mathrm{str}(\Phi)$ is invariant under the full chiral group G and introduces new parameters m_0^2 and α into the $O(p^2)$ euclidean lagrangian for PQChPT, which reads

$$\mathcal{L} = \frac{f^2}{8}\,\mathrm{str}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - \frac{f^2 \mu}{4}\,\mathrm{str}\left(\hat{M}\Sigma + \hat{M}\Sigma^\dagger\right) + \frac{m_0^2}{6}\,\Phi_0^2 + \frac{\alpha}{6}\left(\partial_\mu \Phi_0\right)\left(\partial_\mu \Phi_0\right). \tag{3}$$

We define

$$M_{ij}^2 \equiv \mu(m_i + m_j) \tag{4}$$

(for $i = j = n-N+1, \cdots, n$, this simplifies to $M_{ii}^2 = 2\mu m_S \equiv M_{SS}^2$). It is instructive to display the two-point functions for the neutral mesons (*i.e.* the diagonal fields Φ_{ii}) explicitly. In the diagonal basis with states Φ_{ii}, $i = 1, \cdots, 2n-N$, corresponding to $q_1\overline{q}_1$, $q_2\overline{q}_2, \cdots$, and their ghost-quark counterparts, these two-point functions are, in momentum space [3],

$$G_{i,j}(p) = \frac{\delta_{ij}\epsilon_i}{p^2 + M_{ii}^2} - \frac{1}{(3+N\alpha)} \frac{(m_0^2 + \alpha p^2)(p^2 + M_{SS}^2)}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)(p^2 + m_{\eta'}^2)}\,, \tag{5}$$

where

$$\epsilon_i = \begin{cases} +1, & \text{for } 1 \le i \le n \\ -1, & \text{for } n+1 \le i \le 2n-N \end{cases}\,, \tag{6}$$

and

$$m_{\eta'}^2 = \frac{M_{SS}^2 + N m_0^2/3}{1 + N\alpha/3}\,, \tag{7}$$

which is the square of the pole mass for the Φ_0 two-point function, as can easily be verified from Eq. (5). It is equal to the square of the "η'" meson mass in the $SU(N)$ theory constructed only from the unquenched quarks [3]. It does not vanish in the chiral limit $m_S \to 0$.

A simplification occurs in partially quenched QCD. For a correlation function in which only k out of the $n-N$ valence quarks are on the external lines, the rest of the $n-N-k$

valence quarks do not contribute at all, and therefore this correlation function does not depend on their masses. In most of this paper, we will only consider quantities involving at most two valence quarks, *i.e.* we will only consider valence quarks q_i with $i = 1, 2$.

Compared to quenched ChPT (QChPT), PQChPT introduces an extra mass scale M_{SS}. We can therefore consider various possible "expansion schemes." Let Λ_p be the cutoff of the partially quenched theory. The first scheme, adopted in ref. [4], takes $M_{11},\ M_{22},\ M_{SS} < m_{\eta'} \sim \Lambda_p$. This corresponds to the usual set-up for power counting in ChPT, in which the η' is not a Goldstone boson but instead a "heavy" particle, due to the presence of the scale m_0, which does not vanish in the chiral limit. In this scheme, the field Φ_0 can be integrated out, absorbing dependence on the parameters α and m_0^2 into $O(p^4)$ terms in the chiral lagrangian. This can be illustrated by considering for instance the two-point function $G_{1,1}$ (*cf.* Eq. (5)). We first write the second term in $G_{1,1}$ as

$$-\frac{m_0^2 + \alpha p^2}{3 + N\alpha} \left(\frac{m_{\eta'}^2 - M_{SS}^2}{(m_{\eta'}^2 - M_{11}^2)^2} \left[\frac{1}{p^2 + M_{11}^2} - \frac{1}{p^2 + m_{\eta'}^2} \right] + \frac{M_{SS}^2 - M_{11}^2}{m_{\eta'}^2 - M_{11}^2} \frac{1}{(p^2 + M_{11}^2)^2} \right) . \quad (8)$$

For large $m_{\eta'}$, the expansion of the term containing the pole $1/(p^2 + m_{\eta'}^2)$ in powers of $M_{11}^2/m_{\eta'}^2$ and $M_{SS}^2/m_{\eta'}^2$ generates contact terms which can be absorbed into $O(p^4)$ coefficients of the chiral lagrangian. Similarly expanding the remaining terms in Eq. (8) gives the leading terms

$$-\frac{1}{N} \left(\frac{1}{p^2 + M_{11}^2} + \frac{M_{SS}^2 - M_{11}^2}{(p^2 + M_{11}^2)^2} \right) . \quad (9)$$

The poles in this expression, which no longer depends on α and m_0^2, lead to nonanalytic terms at one loop. For details and further discussion, see ref. [4].

In QChPT, there is no heavy particle with a mass determined by the parameter m_0^2, as can been seen by setting $N = 0$ in Eq. (5). (The Φ_0 two-point function vanishes for $N = 0$ [8].) The second term in Eq. (5) is proportional to $m_0^2 + \alpha p^2$, and has a double pole at the Goldstone meson masses M_{ii}^2. (Of course, no M_{SS} dependence remains.) Hence, this term needs to be kept in the quenched approximation, and the decoupling as described

above does not occur. It turns out that, in order to make sense of the chiral expansion, we need to assume that the parameter $(m_0^2/3)/(4\pi f_\pi)^2$ is small, in addition to the usual requirement that $M_{ii}^2/(4\pi f_\pi)^2$ be small [8,9]. QChPT therefore corresponds to a second "scheme" in which $M_{11},\ M_{22},\ m_0 < \Lambda_p$ and $M_{SS} > \Lambda_p$ which corresponds to freezing out sea-quark loop effects, effectively setting $N = 0$. We note that in the unquenched theory $(m_0^2/3)/(4\pi f_\pi)^2 \approx 0.09$ (where $f_\pi = 132$ MeV is the physical pion decay constant). It is believed that the value of m_0 is not very different in the quenched theory [1].

Since the partially quenched theory "interpolates" between the quenched and unquenched theories, it is natural to consider a third scheme, which we will now explain. First, as in ref. [4], we take $M_{11},\ M_{22},\ M_{SS} < \Lambda_p$, and $M_{11},\ M_{22} < m_{\eta'}$. However, we would like to leave the ratio of M_{SS} and m_0 arbitrary. Let us consider the possibilities. If $M_{SS} < m_{\eta'}$ we can systematically study the effective theory for mesons with masses $M_{11},\ M_{22}$ and M_{SS} by integrating out the η', because now $M_{11},\ M_{22},\ M_{SS} < m_{\eta'}$, as in ref. [4]. If, on the contrary, M_{SS} is of order m_0 or larger, this would be incorrect. Now, we can only systematically investigate the effective theory for mesons with masses $M_{11},\ M_{22}$ and M_{SS}, if we keep the η' in the effective theory, and, as in the completely quenched case, assume that $m_0 < \Lambda_p$.

In partially quenched lattice computations, often M_{SS} is fairly large, and we may well be in the situation that indeed M_{SS} is of order m_0. Therefore, in this paper, we will be interested in calculating the nonanalytic dependence on the quark masses $M_{11},\ M_{22},\ M_{SS}$ for various quantities at one loop, taking $M_{11},\ M_{22},\ M_{SS},\ m_0 < \Lambda_p$ and $M_{11},\ M_{22} < m_{\eta'}$. Note that we can still consider the case that $M_{SS} < m_0$ if $M_{11},\ M_{22} < m_0$, which leads us back to the assumptions made in ref. [4]. Indeed, when we expand our results for meson masses and decay constants in $M_{SS}/m_{\eta'}$, we obtain those of ref. [4].* On the other hand, if we take M_{SS} large and expand in M_{11}/M_{SS}, M_{22}/M_{SS} and m_0/M_{SS}, we obtain the

* Due to a difference in normalization, our tree-level weak decay constant f is by a factor $\sqrt{2}$ larger than that of refs. [4,9]. This should be taken into account before comparing results.

quenched (*i.e.* $N=0$) results of refs. [8,9,13].

Before we present our results, we should address one more issue. In our scheme, there will be one-loop contributions proportional to $\log m_{\eta'}^2$ coming from Φ_0 tadpoles. For values of the meson masses which are small relative to $m_{\eta'}$, such terms can be absorbed into the $O(p^4)$ coefficients (after expanding in $M_{ii}/m_{\eta'}$). However, this is not true for M_{SS} when M_{SS} is of order m_0 or larger. In this case, the dependence of Φ_0 tadpoles on M_{SS} will be complicated. Moreover, the contributions from Φ_0 tadpoles will depend on other η' coupling constants (analogous to m_0^2 and α) which are basically unknown. (We did not include these couplings in Eq. (3); for a more complete expression, see refs. [8,3].)

Therefore, in this paper, we will take the point of view that we keep M_{SS} fixed, and calculate the nonanalytic dependence on the valence-quark masses. This, then, allows us to ignore contributions from Φ_0 tadpoles, which we will do in the rest of this paper. It does not affect the coefficients of the other chiral logarithms. Note that we do not assume the ratios of the valence-quark masses and the sea-quark mass to be small; our scheme allows for arbitrary values of these ratios.

3. Chiral condensate, masses and weak decay constants

We list the one-loop expressions for the condensate, meson masses and decay constants for a theory with nondegenerate valence-quark masses m_1 and m_2 and sea-quark mass m_S in terms of the bare parameters f, M_{ij}^2, m_0^2 and α, including only nonanalytic terms (with $M_{iS}^2=(M_{ii}^2+M_{SS}^2)/2$, *cf.* Eq. (4)):

$$\begin{aligned}
[m_i\langle\overline{q}_i q_i\rangle]_{1-\mathrm{loop}} &= -\frac{M_{ii}^2 f^2}{4}\Bigg(1-2N\frac{M_{iS}^2}{(4\pi f)^2}\log\frac{M_{iS}^2}{\Lambda_p^2} \\
&\qquad -\frac{2}{3(4\pi f)^2}\left[\mathcal{M}^2-AM_{ii}^2+\left(\mathcal{M}^2-2AM_{ii}^2\right)\log\frac{M_{ii}^2}{\Lambda_p^2}\right]\Bigg) \\
&= -\frac{1}{4}[M_{ii}^2]_{1-\mathrm{loop}}[f_{ii}^2]_{1-\mathrm{loop}}\ ,
\end{aligned} \tag{10}$$

$$\left[M_{12}^2\right]_{1-\text{loop}} = M_{12}^2\Bigg(1 + \frac{2}{3(4\pi f)^2}\Bigg[\frac{M_{11}^2\left(\mathcal{M}^2 - AM_{11}^2\right)}{M_{22}^2 - M_{11}^2}\log\frac{M_{11}^2}{\Lambda_p^2}$$
$$- \frac{M_{22}^2\left(\mathcal{M}^2 - AM_{22}^2\right)}{M_{22}^2 - M_{11}^2}\log\frac{M_{22}^2}{\Lambda_p^2}\Bigg]\Bigg)\ , \tag{11}$$

$$[f_{12}]_{1-\text{loop}}/f = 1 - \frac{N}{2(4\pi f)^2}\left(M_{1S}^2\log\frac{M_{1S}^2}{\Lambda_p^2} + M_{2S}^2\log\frac{M_{2S}^2}{\Lambda_p^2}\right) \tag{12}$$
$$+ \frac{1}{3(4\pi f)^2}\left(\mathcal{M}^2\left(\frac{1}{2\epsilon}\log\frac{1+\epsilon}{1-\epsilon} - 1\right) + AM_{12}^2\left(1 - \frac{1-\epsilon^2}{2\epsilon}\log\frac{1+\epsilon}{1-\epsilon}\right)\right)\ ,$$

where

$$\epsilon \equiv \frac{m_2 - m_1}{m_2 + m_1} = \frac{M_{22}^2 - M_{11}^2}{M_{22}^2 + M_{11}^2}\ . \tag{13}$$

All dependence on m_0^2 and α is embodied in the quantities $\mathcal{M}^2$ and A, with

$$\mathcal{M}^2 \equiv \frac{M_{SS}^2\ m_0^2}{M_{SS}^2 + Nm_0^2/3} = \frac{3y}{1+Ny}\ M_{SS}^2\ , \tag{14}$$

$$A \equiv \frac{\alpha M_{SS}^4 + Nm_0^4/3}{\left(M_{SS}^2 + Nm_0^2/3\right)^2} = \frac{\alpha + 3Ny^2}{(1+Ny)^2}\ , \tag{15}$$

where we introduced the ratio $y \equiv (m_0^2/3)/M_{SS}^2$. The results of ref. [4] (with $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_8 = 0$ in Eqs. (13–20) of ref. [4]) can easily be obtained by expanding $\mathcal{M}^2$ and A in $M_{SS}^2/m_{\eta'}^2$ (or M_{SS}^2/m_0^2), and keeping the leading order terms, $3M_{SS}^2/N$ and $3/N$, respectively. (Subleading terms are of higher order in the chiral expansion, and can be dropped.) This corresponds mathematically to taking the limit $y \to \infty$. Let us comment briefly on this comparison between our results and those of ref. [4]. In our case, we keep the η', whereas in ref. [4] the η' is integrated out. One would therefore in general expect that in order to "match" the two theories, we would need to adjust the bare parameters. For the quantities considered here, it turns out that, at one loop, all nontrivial adjustments come from Φ_0-tadpole contributions. However, we did not have to, and hence did not, include such contributions, as was explained in section 2. We conclude that, for the Goldstone meson masses and decay constants, no adjustment is needed.

The results of QChPT [8,9] can be obtained by taking $N = 0$, for which $\mathcal{M}^2 = m_0^2$ and $A = \alpha$.

With degenerate quark masses $m_V \equiv m_1 = m_2$ in Eq. (11) and Eq. (12), we obtain

$$M_{VV}^2 \equiv \left[M_{11}^2\right]_{1-\mathrm{loop}} = M_{11}^2 \left(1 - \frac{2}{3(4\pi f)^2}\left[\mathcal{M}^2 - AM_{11}^2 + \left(\mathcal{M}^2 - 2AM_{11}^2\right)\log\frac{M_{11}^2}{\Lambda_p^2}\right]\right) , \tag{16}$$

and

$$\frac{f_{VV}}{f} \equiv \frac{[f_{11}]_{1-\mathrm{loop}}}{f} = 1 - N\frac{M_{1S}^2}{(4\pi f)^2}\log\frac{M_{1S}^2}{\Lambda_p^2} . \tag{17}$$

As in QChPT, the ratio of M_{VV}^2 to its tree-level value, M_{VV}^2/M_{11}^2, is singular, while f_{VV} is regular in the chiral limit $m_V \to 0$. Moreover, f_{VV} does not depend on m_0^2 and α.

We can define M_{VS}^2 and f_{VS} to one loop from Eqs. (11,12) by replacing m_1 by m_V and m_2 by m_S. In the chiral limit $m_V \to 0$, keeping m_S fixed, we have $\epsilon \to 1$ and the nonanalytic functions multiplying $\mathcal{M}^2$ and A in the expression for f_{VS}, *cf.* Eq. (12), diverge. It is obvious from Eq. (11), that the ratio of M_{VS}^2 to its tree-level value is regular in the chiral limit, $m_V \to 0$.

Of course, all quantities considered here also receive analytic contributions from $O(p^4)$ terms in the chiral lagrangian. Since they are obtained from tree-level diagrams with $O(p^4)$ vertices, they do not explicitly depend on the parameters m_0^2 and α, and therefore are identical to those reported in ref. [4], to which we refer for their explicit form. In ref. [4] it was pointed out that the quantity

$$M_{VV}^2 - M_{V'S}^2 \quad \text{with} \quad m_{V'} = 2m_V - m_S , \tag{18}$$

is independent of $O(p^4)$ coefficients.

Another quantity which is independent of $O(p^4)$ coefficients, first introduced in ref. [14] for the quenched case, and also considered in ref. [4] for the partially quenched case at

$y = \infty$, is

$$\left[\frac{f_{12}}{\sqrt{f_{11}f_{22}}}\right]_{1-\text{loop}} = 1+\frac{1}{3(4\pi f)^2}\left(\mathcal{M}^2\left(\frac{1}{2\epsilon}\log\frac{1+\epsilon}{1-\epsilon}-1\right)+AM_{12}^2\left(1-\frac{1-\epsilon^2}{2\epsilon}\log\frac{1+\epsilon}{1-\epsilon}\right)\right). \tag{19}$$

For any m_S, this quantity diverges when $m_1 \to 0$ with m_2 fixed (*i.e.* for $\epsilon \to 1$). Note that $[f_{12}]_{1-\text{loop}}$ is an even function of ϵ (because of symmetry under the interchange $m_1 \leftrightarrow m_2$), so that $[\, f_{12}/\sqrt{f_{11}f_{22}}\,]_{1-\text{loop}} = 1 + O(\epsilon^2)$ for $\epsilon \to 0$.

From Eqs. (11,12,16), we see that the coefficients of the chiral logarithms of the valence-quark mass depend on the ratio y of the parameter m_0^2 and the sea-quark mass, through the quantities $\mathcal{M}^2$ and A (*cf.* Eqs. (14,15)). From (partially) quenched lattice data, it is estimated that $m_0^2/3$ presumably has a value $m_K^2/2 \lesssim m_0^2/3 \lesssim m_K^2$ ($m_K = 496$ MeV is the physical kaon mass) [1,15]. Typical lattice computations have $N = 2$ and $m_K^2 \lesssim M_{SS}^2 \lesssim 2m_K^2$. These values of m_0^2 and M_{SS}^2 correspond to y ranging from $y \approx 1/4$ to $y \approx 1$. This leads to $\mathcal{M}^2/M_{SS}^2 = 1/2$ for $y = 1/4$ and to $\mathcal{M}^2/M_{SS}^2 = 1$ for $y = 1$. For $y \to \infty$ one obtains $\mathcal{M}^2/M_{SS}^2 = 3/2$. This shows that for relatively heavy sea quarks, there is a clear dependence of the coefficient of the chiral logarithms on m_0^2. (Experience with quenched lattice data [1] indicates that it is hard to fit the chiral logarithms reliably, partially because of the "competition" of $O(p^4)$ coefficients. This may make it difficult to see the y dependence of the chiral logarithms in practice.)

The quantity A also has an effect on the coefficients of the chiral logarithms, in particular for values of the valence-quark mass of order of the sea-quark mass. Taking again $N = 2$, we find $A = (8\alpha + 3)/18$ for $y = 1/4$ and $A = (\alpha + 6)/9$ for $y = 1$, while $A = 3/2$ for $y \to \infty$. It is clear that A is more sensitive to the value of α for smaller values of y. We note that for $m_V/m_S = 1$, $M_{11}^2/m_{\eta'}^2 = 1/(1 + Ny)$ (for $\alpha = 0$), so that our results may not be reliable for smaller values of y.

4. B_K and $K^+ \to \pi^+\pi^0$ decay

In this section, we will generalize earlier quenched one-loop calculations for B_K [9,16,13] and $K^+ \to \pi^+\pi^0$ [13] to the partially quenched case. In the following, u, d and s denote valence quarks with masses m_1, m_1 and m_2 respectively. The kaon B-parameter B_K is defined as (with M_K the mass of the $d\bar{s}$ meson, *i.e.* at tree level $M_K = M_{12}$)

$$B_K = \frac{\langle \overline{K}^0 | (\bar{s}d\bar{s}d)_{LL} | K^0 \rangle}{\frac{8}{3} f_K^2 M_K^2} , \tag{20}$$

in which the four-quark operator is defined by

$$(\bar{q}_i q_j \bar{q}_k q_l)_{LL} = (\bar{q}_{iL} \gamma^\mu q_{jL})(\bar{q}_{kL} \gamma_\mu q_{lL}) , \tag{21}$$

where $q_L = \frac{1}{2}(1-\gamma_5)q$ is a left-handed quark field. The denominator in Eq. (20) is the matrix element $\langle \overline{K}^0 | (\bar{s}d\bar{s}d)_{LL} | K^0 \rangle$ evaluated by vacuum saturation. The $O(p^2)$ weak-interaction operator in ChPT corresponding to $(\bar{s}d\bar{s}d)_{LL}$, which is the $\Delta S = 2$ component of a 27-plet under $SU(3)_L$, is

$$O' = \alpha_{27} \, t^{ij}_{kl} (\Sigma \partial^\mu \Sigma^\dagger)_i{}^k (\Sigma \partial_\mu \Sigma^\dagger)_j{}^l , \tag{22}$$

where the tensor t^{ij}_{kl} is defined by setting

$$t^{22}_{33} = 1 , \tag{23}$$

while all other components are equal to zero. The parameter α_{27} is the only new $O(p^2)$-operator coefficient; its value is determined by QCD dynamics. At tree level, one obtains [17]

$$\langle \overline{K}^0 | O' | K^0 \rangle = \frac{8\alpha_{27}}{f^2} M_K^2 , \tag{24}$$

and therefore,

$$B_K = \frac{3\alpha_{27}}{f^4} \equiv B \tag{25}$$

($f_K = f$ at tree level). Since the partially quenched theory is different from unquenched QCD, the partially quenched value of the coefficient α_{27} is in principle different from the QCD value. We will make this explicit by using a subscript or superscript p to denote bare parameters of the partially quenched theory, specifically, α_{27}^p, B^p and f_p.

The partially quenched one-loop result for B_K^p with non-degenerate quark masses, keeping only nonanalytic terms, is

$$
\begin{aligned}
&B_K^p = B^p \times \qquad\qquad (26)\\
&\left(1+\frac{M_{12}^2}{(4\pi f_p)^2}\left(-2(3+\epsilon^2)\log\frac{M_{12}^2}{\Lambda_p^2}-(2+\epsilon^2)\log\left(1-\epsilon^2\right)-3\epsilon\log\frac{1+\epsilon}{1-\epsilon}\right)\right.\\
&\quad+\frac{2}{3(4\pi f_p)^2}\left[\mathcal{M}^2\left(\frac{2-\epsilon^2}{2\epsilon}\log\frac{1+\epsilon}{1-\epsilon}-2\right)\right.\\
&\qquad\qquad\left.\left.+AM_{12}^2\left(2+\epsilon^2-\frac{1-2\epsilon^2-\epsilon^3}{\epsilon}\log\frac{1+\epsilon}{1-\epsilon}+2\epsilon^2\log\left(\frac{M_{12}^2\left(1-\epsilon\right)}{\Lambda_p^2}\right)\right)\right]\right) ,
\end{aligned}
$$

where ϵ is defined in Eq. (13). It can easily be seen that Eq. (26) does not depend on m_0^2 and α in the case of degenerate quark masses ($\epsilon = 0$), just as in QChPT. Actually, apart from changing the values of B and f to their (partially) quenched values, (partial) quenching does not introduce any change in the nonanalytic one-loop corrections of B_K in the degenerate case. For the quenched case this was already discussed in ref. [9], and the fact that this is also true in PQChPT does not come as a surprise, since PQChPT is "in between" ChPT and QChPT (see also the discussion of $K^+ \to \pi^+\pi^0$ below). In the nondegenerate case, the partially quenched result follows from the quenched result by replacing $m_0^2 \to \mathcal{M}^2$, $\alpha \to A$. For a discussion of contributions from $O(p^4)$ coefficients, see ref. [13].

The $\Delta S = 1,\ \Delta I = 3/2\ \ K^+ \to \pi^+\pi^0$ decay amplitude is proportional to the weak matrix element

$$
\langle\pi^+\pi^0|\left(\bar{s}d\bar{u}u+\bar{s}u\bar{u}d-\bar{s}d\bar{d}d\right)_{LL}|K^+\rangle\ . \qquad (27)
$$

The four-fermion operator is the $\Delta I = 3/2$ component of the same 27-plet that also contains

the operator $(\bar{s}d\bar{s}d)_{LL}$ (Eq. (20)). To $O(p^2)$ in ChPT, the operator is represented by

$$O_4 = \alpha_{27}\, r^{ij}_{kl} (\Sigma\partial^\mu\Sigma^\dagger)_i^{\ k} (\Sigma\partial_\mu\Sigma^\dagger)_j^{\ l} \ , \tag{28}$$

where the tensor r^{ij}_{kl} has nonzero components

$$\begin{aligned} r^{21}_{31} = r^{12}_{13} = r^{12}_{31} = r^{21}_{13} &= \frac{1}{2} \ , \\ r^{22}_{32} = r^{22}_{23} &= -\frac{1}{2} \end{aligned} \tag{29}$$

(all other components vanish). The parameter α_{27} is the same as in Eq. (22). The aim is then to calculate the matrix element $\langle\pi^+\pi^0|O_4|K^+\rangle$ to one loop.

The lattice determination of $\langle\pi^+\pi^0|O_4|K^+\rangle$ was compared with its real-world value in great detail at one loop in ChPT and QChPT in ref. [13], and here we will only discuss what is new in the partially quenched case.

All attempts to compute this matrix element on the lattice have been restricted to the mass-degenerate, quenched theory, and moreover, all mesons are taken to be at rest [18,19,20]. The operator O_4 then inserts energy, implying that the values thus obtained are unphysical. All these systematic errors can be studied in ChPT. Deviations due to the choice of unphysical masses and momenta already shows up at tree level [18], while all three systematic effects (including quenching) lead to one-loop contributions different from those calculated for physical masses and momenta in the unquenched theory [13]. In addition, at one loop one finds that there are power-like finite-volume corrections, which were also calculated in ref. [13].

The one-loop result for this unphysical matrix element in a finite volume L^3 (with periodic boundary conditions) and with $m_1 = m_2 = m_V$ (hence $M_\pi = M_K$, with M_π the mass of the $u\overline{d}$ meson) for the partially quenched theory is, keeping only non-analytic corrections,

$$\begin{aligned} \langle\pi^+\pi^0|O_4|K^+\rangle^p_{unphys} = \frac{24i\alpha^p_{27}M_\pi^2}{\sqrt{2}f_p^3}\Bigg(1 - N\frac{M_{1S}^2}{(4\pi f_p)^2}\log\frac{M_{1S}^2}{\Lambda_p^2} \\ + \frac{M_{11}^2}{(4\pi f_p)^2}\left[-3\log\frac{M_{11}^2}{\Lambda_p^2} + F(M_{11}L)\right]\Bigg) \ , \end{aligned} \tag{30}$$

up to corrections vanishing faster than any power of L^{-1}, where the function F is given by

$$F(x) = \frac{17.827}{x} + \frac{12\pi^2}{x^3} \ . \tag{31}$$

We note that the chiral logarithm due to the sea-quark loops does not diverge in the chiral limit $m_V \to 0$. Note also that the result Eq. (30) does not depend on the parameters m_0^2 and α. For a discussion of contributions from $O(p^4)$ coefficients, see ref. [13].

The above result can also be derived from the "quark flow picture" [7,9], and the results of ref. [13]. First, take all quark masses equal, including the sea-quark mass. The quenched and unquenched results of ref. [13] then are the special cases obtained from Eq. (30) by setting $N = 0$ and $N = 3$ respectively. The difference is due to the sea-quark loops which are present in the unquenched case, but not in the quenched case. Therefore, if we now have N instead of 3 sea quarks, one obtains the correct result by multiplying the difference between the unquenched and quenched results by $N/3$, yielding the first chiral logarithm in Eq. (30). But now we also identified which of the logarithms is due to sea-quark loops (the term proportional to N), and we conclude that we obtain the partially quenched result for $m_S \neq m_V$ by replacing $M_{11}^2 \to M_{1S}^2$ in the term linear in N. (Because of the structure of O_4 and the $O(p^2)$ vertices from the effective lagrangian Eq. (3), there can be at most one sea-quark on the loop. Therefore, the result does not depend on M_{SS} and the parameters m_0 and α, *cf.* Eq. (5).) Of course, one should keep in mind that the parameters f_p and α_{27}^p depend on N.

The power-like finite-volume corrections in Eq. (30) are independent of N. This follows from the fact that they originate from diagrams which do not contain sea-quark loops [13,21].

5. Numerical examples

In this section, we will give some numerical examples of the differences between the one-loop estimates for B_K and $\langle \pi^+ \pi^0 | O_4 | K^+ \rangle$ calculated in the partially quenched theory

and in the "real world." For $\langle\pi^+\pi^0|O_4|K^+\rangle$ we will also take the other systematic effects discussed in section 4 into account. We will take values for the lattice parameters typical of those used in recent numerical computations (for partially quenched results for B_K, see refs. [22,23]; for $\langle\pi^+\pi^0|O_4|K^+\rangle$ we are not aware of any lattice data).

The general strategy for our estimates is the same as in ref. [13]. We will set all $O(p^4)$ coefficients to zero. We choose the cutoffs Λ (for the full theory) and Λ_p (for the partially quenched theory) to be 1 GeV or 770 MeV, independent of one another. The sensitivity under a change in Λ and Λ_p is taken as an indication of the systematic error associated with our ignorance of the values of $O(p^4)$ coefficients. For the real-world values of f_π, m_π and m_K we will use $f_\pi = 132$ MeV, $m_\pi = 136$ MeV and $m_K = 496$ MeV.

A. B_K

B_K in the full and partially quenched theories can be related by using Eq. (26) above and Eq. (36) of ref. [13]:

$$\frac{B_K^{phys}}{B_K^p} = \frac{\alpha_{27}}{\alpha_{27}^p}\left(\frac{f_p}{f}\right)^4 P \ , \tag{32}$$

where

$$P = \frac{1+H}{1+\tilde{H}} \ , \tag{33}$$

with $H = 0.724$ (for $\Lambda = 1$ GeV) and $H = 0.417$ (for $\Lambda = 770$ MeV) is the numerical value of the relative one-loop correction for B_K in the real world, and $\tilde{H}$ is the relative one-loop correction for the partially quenched theory in Eq. (26). To one-loop accuracy, M_{12}^2 can be replaced by M_K^2. The factor P incorporates all one-loop corrections ($P = 1$ at tree level).

Since the ratios f_p/f and $\alpha_{27}/\alpha_{27}^p$ cannot be determined within ChPT, we will arbitrarily set them equal to one. This constitutes one of the major uncertainties of our method. Furthermore, we will take $\alpha = 0$ for simplicity. In Table 1, we list the numerical values of P for different combinations of cutoffs, choosing $f_p = f_\pi$, $\epsilon = 1/2$, $M_{SS} = m_K$, $N = 2$, for various choices of the parameters $y = (m_0^2/3)/M_{SS}^2$ and M_K^2.

M_K^2	y	$P_{(1)}^{(1)}$	$P_{(0.77)}^{(0.77)}$	$P_{(0.77)}^{(1)}$	$P_{(1)}^{(0.77)}$
	0.5	1.02	0.98	1.19	0.84
0.2	1	1.02	0.98	1.19	0.84
	∞	1.01	0.96	1.17	0.83
	0.5	0.99	1.13	1.37	0.81
0.4	1	0.98	1.11	1.35	0.81
	∞	0.96	1.05	1.28	0.79
	0.5	1.11	1.73	2.10	0.92
0.6	1	1.09	1.64	1.99	0.90
	∞	1.03	1.42	1.73	0.84

Table 1. The factor P, Eq. (33), for different values of M_K^2, y and different combinations of Λ and Λ_p. Other parameters are fixed at $f_p = f_\pi$, $\epsilon = 1/2$, $M_{SS} = m_K$ and $N = 2$. The superscript on P denotes Λ in GeV*; the subscript on P denotes Λ_p in* GeV*. M_K^2 is in* GeV^2*.*

For $M_K^2 = 0.6$ GeV^2, values for P as high as 2 are obtained, for $\Lambda_p = 770$ MeV. However, for this cutoff, the meson mass is probably too large for PQChPT to be reliable. Also, for $M_K^2 = 0.6$ GeV^2 and $y = 0.5$, $M_K^2/m_{\eta'}^2$ is of order one. We will therefore concentrate on the two lower masses in the following discussion. For these values, we see that P never differs from one by more than about 10%, if we choose $\Lambda = \Lambda_p$, which is equivalent to the assumption that the bare parameters of the full and partially quenched theories are equal (this may not be unreasonable for $N = 2$). However, if we take the two cutoffs unequal, corrections can be as large as $20 - 30\%$. The results are fairly insensitive to changes in the value of y.

B. $K^+ \to \pi^+\pi^0$

The $K^+ \to \pi^+\pi^0$ matrix elements in the full and partially quenched theories can be related by using Eq. (30) above and Eqs. (43,87) of ref. [13] (replacing M_{11}^2 by M_π^2 and M_{1S}^2

by M_{VS}^2):

$$\langle\pi^+\pi^0|O_4|K^+\rangle_{phys} = W\ \frac{\alpha_{27}}{\alpha_{27}^p}\ \left(\frac{f_p}{f}\right)^3\ \frac{m_K^2 - m_\pi^2}{2M_\pi^2}\ \langle\pi^+\pi^0|O_4|K^+\rangle_{unphys}^p\ , \tag{34}$$

with

$$W = \frac{1+U}{1 - N\frac{M_{VS}^2}{(4\pi f_p)^2}\log\frac{M_{VS}^2}{\Lambda_p^2} + \frac{M_\pi^2}{(4\pi f_p)^2}\left[-3\log\frac{M_\pi^2}{\Lambda_p^2} + F(M_\pi L)\right]}\ , \tag{35}$$

where $U = 0.0888$ (for $\Lambda = 1$ GeV) and $U = -0.0146$ (for $\Lambda = 770$ MeV) is the numerical value of the relative one-loop correction in the real world. At tree level, $W = 1$.

We list in Table 2a the numerical values of W for different combinations of the cutoffs Λ and Λ_p, and for $f_p = f_\pi$, $M_{SS} = m_K$, $N = 2$, several values of M_π^2 and a fixed volume L^3 such that $M_\pi L = 6$ for $M_\pi^2 = 0.2$ GeV^2. Here we may use $M_{VS}^2 = (M_\pi^2 + M_{SS}^2)/2$. In Table 2b we list values of W for infinite volume, with all other parameters the same as in Table 2a.

We see from Table 2 that one-loop corrections are always rather large, even for relatively small meson masses, and that sensitivity to the values of the cutoffs is significant. (For infinite volume, the corrections are not quite as big, *cf.* Table 2b.) This casts some doubt on the accuracy of one-loop ChPT in estimating the factor W, and one would expect that two-loop contributions are not small. As in the quenched case, the "correction factor" W is always substantially smaller than one. For a much more detailed discussion of uncertainties inherent to our estimates of such "correction factors," see ref. [13].

As can be seen from Eq. (30), the partially quenched result is closer to the unquenched case (for which $N = 3$, $M_{1S} = M_{11}$) than to the quenched case. The large deviations from the tree-level value $W = 1$ are mostly due to the other systematic effects. For instance, in the unquenched theory, we find $W = 0.56$ ($\Lambda = 1$ GeV) and $W = 0.57$ ($\Lambda = 770$ MeV) for $M_\pi^2 = 0.2$ GeV^2.

M_π^2	$W_{(1)}^{(1)}$	$W_{(0.77)}^{(0.77)}$	$W_{(0.77)}^{(1)}$	$W_{(1)}^{(0.77)}$
0.2	0.59	0.60	0.66	0.53
0.4	0.54	0.60	0.66	0.49
0.6	0.55	0.66	0.73	0.49

Table 2a. The factor W for different values of M_π^2 and different combinations of Λ and Λ_p. Other parameters are fixed at: $f_p = f_\pi$, $M_{SS} = m_K$, $N = 2$ and $M_\pi L = 6$ for $M_\pi^2 = 0.2$. The superscript on W denotes Λ in GeV*; the subscript on W denotes Λ_p in* GeV*. M_π^2 is in* GeV^2.

M_π^2	$W_{(1)}^{(1)}$	$W_{(0.77)}^{(0.77)}$	$W_{(0.77)}^{(1)}$	$W_{(1)}^{(0.77)}$
0.2	0.68	0.71	0.78	0.62
0.4	0.65	0.75	0.83	0.59
0.6	0.68	0.90	0.99	0.62

Table 2b. The factor W for different values of M_π^2 and different combinations of Λ and Λ_p. Other parameters are fixed at: $f_p = f_\pi$, $M_{SS} = m_K$, $N = 2$ and $M_\pi L = \infty$. The superscript on W denotes Λ in GeV*; the subscript on W denotes Λ_p in* GeV*. M_π^2 is in* GeV^2.

6. Conclusion

In this paper, we applied PQChPT to the calculation of the quark-mass dependence of various quantities to one loop, restricting ourselves to degenerate sea-quark masses. For Goldstone boson masses and decay constants, we extended results obtained earlier in ref. [4]: we investigated the sensitivity of the chiral logarithms to the singlet part of the η' mass, m_0. Since the "remnants" of the η' are an essential part of the fully quenched theory, this is a natural question to ask in the partially quenched theory. We found that the coefficients of the chiral logarithms for the Goldstone meson masses are sensitive to the value of m_0 for

typical values of the sea-quark mass. The decay constants depend on m_0 to a lesser extent, which is related to the fact that in the limit of degenerate quark masses, the axial current does not couple to the η'.

We also calculated one-loop contributions to the chiral condensate, B_K and the $K^+ \to \pi^+\pi^0$ decay amplitude, extending earlier work in QChPT [9,16,13] to the partially quenched case. For $K^+ \to \pi^+\pi^0$ we only considered the case of degenerate valence-quark masses.

We considered some numerical examples of the comparison between B_K and the $K^+ \to \pi^+\pi^0$ decay amplitude in the real world and in partially quenched QCD with values of the parameters typical of current lattice computations. For B_K, we found that, for small enough meson masses, the "correction factor" may be very close to one, but with an uncertainty which could be as large as $20 - 30\%$. For $K^+ \to \pi^+\pi^0$, we also took into account that typical lattice computations are done at unphysical (degenerate) valence-quark masses and external momenta, and we included the leading finite-volume corrections. We found that the "correction factor" is always much smaller than one (of order one-half, with large uncertainties). This is mostly due to the unphysical choice of masses and momenta, and not to partial quenching.

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